Chapter 10: Symmetrical Components and Unbalanced Faults

10.1 Introduction

When an unbalanced three-phase fault occurs, we can solve the three-phase circuit using ordinary circuit theory. This is much more numerically complicated than the single-phase circuit normally used in balanced three phase circuits. The degree of difficulty increases with the third power of the system size. For this reason, it is apparent that if we were to solve three different single-phase circuits, it would be numerically simpler than solving the one three-phase circuit in one set of equations.

The purpose of this chapter is to break up the large three-phase circuit into three circuits, each one third the size of the whole system. Next, we solve the three components individually, and then combine the results to obtain the total system response. The two programs abc2sc and sc2abc will transform the system representation from the normal "abc" form to the "sc" or symmetrical components form (and the other way around.) Three other programs are developed, these are:

- Line to ground fault studies: `lgfault(zdata0, zbus0, zdata1, zbus1, zdata2, zbus2, V)`
- Line to line fault studies: `llfault(zdata1, zbus1, zdata2, zbus2, V)`
- Double line to ground fault studies: `dlgfault(zdata0, zbus0, zdata1, zbus1, zdata2, zbus2, V)`

As can be seen above, these programs use the three components which are developed first, namely the "0", "1" and "2" components. These are often referred to as the zero-sequence, positive-sequence and negative-sequence components, respectively. Some other books use "0", "+" and "−" for these same components.

10.2 Fundamentals of Symmetrical Components

It was Fortescue in 1918 who developed the idea of breaking up asymmetrical three-phase voltages and currents into three sets of symmetrical components. These three basic components are:

1. The positive sequence currents and voltages (known also as the "abc" and often denoted by the superscript "1" or "+", shown on the right. Note this is the regular three-phase balanced phasors we have been dealing with from the start of this course.
(2) The negative sequence currents and voltages (known also as the "acb" and often denoted by the superscript "2" or "−"). These are the regular balanced negative sequence components we have seen previously. Note the sequence of the phasors is the opposite direction from the positive sequence (acb instead of abc.)

(3) The zero sequence components of currents and voltages (often denoted by the superscript "0"). These components are shown on the right. Note that these zero sequence phasors are all in-phase and equal in magnitude.

We will need the vector \( a = 1\angle 120^\circ \), which is a unit vector at an angle of 120 degrees. It is easy to see that \( a^2 = a^* = 1\angle 240^\circ = 1\angle -120^\circ \) and \( a^3 = 1 \). It is also clear that \( 1 + a + a^2 = 0 \).

\[
\begin{align*}
I_a^1 &= I_a^0 \angle 0^\circ = I_a^1 \\
\text{Positive sequence:} & \quad I_b^1 = I_a^0 \angle 240^\circ = a^2 I_a^1 \\
& \quad I_c^1 = I_a^0 \angle 120^\circ = a I_a^1 \\
I_a^2 &= I_a^2 \angle 0^\circ = I_a^2 \\
\text{Negative sequence:} & \quad I_b^2 = I_a^2 \angle 120^\circ = a I_a^2 \\
& \quad I_c^2 = I_a^2 \angle 240^\circ = a^2 I_a^2 \\
I_a^0 &= I_b^0 = I_c^0 \\
\text{Zero sequence:} & \quad I_a^0 \\
\end{align*}
\]

Now consider a three-phase unbalanced set of currents \( I_a \), \( I_b \), and \( I_c \) as shown in the figure to the right. Note that the vectors are arbitrary in length and direction. The same argument would hold if these were voltages instead of currents. These "abc" currents can be expressed in terms of the "se" currents as follows:

\[
\begin{align*}
I_a &= I_a^0 + I_a^1 + I_a^2 \\
I_b &= I_b^0 + I_b^1 + I_b^2 \\
I_c &= I_c^0 + I_c^1 + I_c^2 \\
\end{align*}
\]

And we know that all the currents can be expressed in terms of the "a" components, thus:

\[
\begin{align*}
I_a &= I_a^0 + I_a^1 + I_a^2 \\
I_b &= I_a^0 + a^2 I_a^1 + a I_a^2 \\
I_c &= I_a^0 + a I_a^1 + a^2 I_a^2 \\
\end{align*}
\]
Or in matrix form:

\[
\begin{pmatrix}
I_a \\
I_b \\
I_c
\end{pmatrix} =
\begin{pmatrix}
1 & 1 & 1 \\
1 & a^2 & a \\
1 & a & a^2
\end{pmatrix}
\begin{pmatrix}
I_a^0 \\
I_a^1 \\
I_a^2
\end{pmatrix}
\]

Which can be written as a single matrix equation as: \( I_{abc}^{012} = A I_a^{012} \), where \( A \) is known as the symmetrical components transformation matrix, which transforms the currents \( I_{abc} \) into component currents \( I_a^{012} \) and is given by: \( A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{pmatrix} \). Note that \( A^T = A \), thus the A-matrix is symmetric.

Using matrix inversion we have: \( I_a^{012} = A^{-1} I_{abc} \) and the inverse matrix is given by:

\[
A^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{pmatrix}
\]

And since \( a^2 = a^* \), we see that \( A^{-1} = \frac{1}{3} A^* \). Using the above equation it is seen that:

\[
\begin{pmatrix}
I_a^0 \\
I_a^1 \\
I_a^2
\end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{pmatrix}
\begin{pmatrix}
I_a \\
I_b \\
I_c
\end{pmatrix}
\]

Or in detail:

\[
I_a^0 = \frac{1}{3} (I_a + I_b + I_c)
\]
\[
I_a^1 = \frac{1}{3} (I_a + aI_b + a^2I_c)
\]
\[
I_a^2 = \frac{1}{3} (I_a + a^2I_b + aI_c)
\]

It is noted that \( I_a^0 \) is the sum of the three "abc" components, thus if the abc components are balanced, their sum is zero, hence the zero-sequence component is also zero.

The exact similar expressions are valid for voltages \( V_{abc} \) and \( V_a^{012} \). These equations need not be repeated here.

Often it is understood that the quantities with superscript "0", "1", or "2" will have subscript "a" and the "a" is omitted. Thus instead of \( I_a^0 \) we use \( I^0 \) and instead of \( I_a^1 \) we use \( I^1 \) and so on…

Next we investigate the three phase complex power. We know that

\[
S_{(3b)} = V_{abc}^* I_{abc}^T
\]
The equation for complex power may be expressed in terms of symmetrical components thus:

\[ S_{(30)} = (AV_{a}^{012})^{T} (AI_{a}^{012})^{*} \]

And dropping the subscript "a" these may be written as:

\[ S_{(30)} = (AV^{012})^{T} (AI^{012})^{*} = V^{012T} A^{*} T^{012*} \]

Keeping in mind that "A" is symmetric and \( A^{T} A^{*} = 3 \), we have:

\[ S_{(30)} = 3 \left( V^{012T} I^{012*} \right) \]

\[ = 3V_{0}I_{0}^{*} + 3V_{1}I_{1}^{*} + 3V_{2}I_{2}^{*} \]

The last equation shows that the total unbalanced three-phase power is equal to the sum of the three symmetrical component powers. Note that in the above equations the subscript "a" was dropped from all those terms having superscript "012," "0," "1," or "2."

Please read about the functions defined on page 404: sctm, phasor, abc2sc, sc2abc, zabc2sc, rec2pol, and pol2rec.

**Example 10.1**

Obtain the symmetrical components of a set of unbalanced currents \( I_{a} = 1.6\angle25^{\circ}, \)
\( I_{b} = 1.0\angle180^{\circ}, \) and \( I_{c} = 0.9\angle132^{\circ}. \)

The Matlab program follows:

```matlab
Iabc = [1.6 25
        1.0 180
        0.9 132];
I012 = abc2sc(Iabc);  % Symmetrical components of phase a
I012p = rec2pol(I012) % Converts rectangular phasors into polar form
```

\[
I012p =
\begin{bmatrix}
0.4512 & 96.4529 \\
0.9435 & -0.0550 \\
0.6024 & 22.3157
\end{bmatrix}
\]
Example 10.2

The symmetrical components of a set of unbalanced three-phase voltages are \( V^0 = 0.6 \angle 90^\circ \), \( V^1 = 1.0 \angle 30^\circ \), and \( V^2 = 0.8 \angle -30^\circ \). Obtain the original unbalanced phasors.

The Matlab program follows:

```matlab
V012 = [.6 90 1 30 0.8 -30];
Vabc = sc2abc(V012); % Unbalanced phasors from symmetrical components
Vabcp = rec2pol(Vabc) % Converts rectangular phasors into polar form
```

\[
\begin{array}{c|c|c|c}
\text{Original Phasors} & \text{Polar Form} \\
\hline
1.7088 & 24.1825 & 1.7088 & 155.8175 \\
0.4000 & 90.0000 &
\end{array}
\]
10.3 Sequence Impedances

Each of the sequence currents (or voltages) will encounter a different impedance in the network. In other words, each network component will present a different impedance as seen by each of the sequences. The network components are: loads, transmission lines, generators and transformers. We shall consider the impedances each of these components presents. $Z^1$ is the usual impedance for positive sequence currents, $Z^2$ is impedance for negative sequence currents, and $Z^0$ is the impedance for zero sequence currents.

10.3.1 Sequence Impedances of Y-Connected Loads

The balanced three-phase Y-load is shown in the figure to the right. There is a mutual coupling between the three loads making up the Y-load. The return current through the neutral goes through an impedance $Z_n$. The "abc" voltages to ground are:

\[
\begin{align*}
V_a &= Z_a I_a + Z_m I_b + Z_m I_c + Z_n I_n \\
V_b &= Z_m I_a + Z_s I_b + Z_m I_c + Z_n I_n \\
V_c &= Z_m I_a + Z_m I_b + Z_s I_c + Z_n I_n
\end{align*}
\]
From Kirchhoff's current law, we have \( I_n = I_a + I_b + I_c \). Using this equation above, we have:

\[
\begin{align*}
V_a &= \begin{pmatrix} Z_s + Z_n \\ Z_m + Z_n \\ Z_m + Z_n \end{pmatrix} I_a \\
V_b &= \begin{pmatrix} Z_s + Z_n \\ Z_s + Z_n \\ Z_s + Z_n \end{pmatrix} I_b \\
V_c &= \begin{pmatrix} Z_s + Z_n \\ Z_m + Z_n \\ Z_s + Z_n \end{pmatrix} I_c
\end{align*}
\]

Or in compact form: \( V^{abc} = Z^{abc} I^{abc} \). Using symmetrical components (recall that \( I_{a012} \) is denoted by \( I^{012} \) for simplicity,) this equation is written as: \( A V^{012} = Z^{abc} A I^{012} \) and solving for the voltage we have:

\[
V^{012} = A^{-1} Z^{abc} A I^{012}
\]

Where:

\[
Z^{012} = A^{-1} Z^{abc} A
\]

Using previously derived matrices we have:

\[
Z^{012} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{pmatrix} \begin{pmatrix} Z_s + Z_n \\ Z_m + Z_n \\ Z_m + Z_n \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} Z_s + 3Z_n + 2Z_m & 0 & 0 \\ 0 & Z_s - Z_m & 0 \\ 0 & 0 & Z_s - Z_m \end{pmatrix}
\]

Performing the above matrix multiplications, we have:

\[
Z^{012} = \begin{pmatrix} Z_s + 3Z_n + 2Z_m & 0 & 0 \\ 0 & Z_s - Z_m & 0 \\ 0 & 0 & Z_s - Z_m \end{pmatrix}
\]

And if there is no mutual coupling between the legs of the Y-load, we have:

\[
Z^{012} = \begin{pmatrix} Z_s + 3Z_n & 0 & 0 \\ 0 & Z_s & 0 \\ 0 & 0 & Z_s \end{pmatrix}
\]

N.B. The impedance matrix has non-zero elements only on the diagonal, therefore for balanced load the three sequences are independent. This means currents in one sequence will produce voltages in that sequence only! This is a very important property since it allows us to analyze each sequence network on a per-phase basis.

### 10.3.2 Sequence Impedances of Transmission Lines

The transmission lines present the same geometry to the positive as well as the negative sequence currents, thus for lines we have \( Z^1 = Z^2 \). However, the zero-sequence impedance is much larger (in general) and is given by: \( X^0 = X^1 + 3X^n \) where

\[
X^n = 2\pi f \left( 0.21 \ln \frac{D_n}{D} \right) \text{m} \ \Omega / \text{km}
\]

Here \( D_n \) is the distance between the line and the
neutral (usually earth or ground,) and $D$ is the distance between the three lines (assuming they are equidistant.)

### 10.3.3 Sequence Impedances of Synchronous Machine

Since the machine has rotation, the inductances seen will differ depending on the direction of rotation of the armature currents. Thus the negative sequence currents will appear to have twice line frequency. However, zero sequence currents will rotate with the windings, thus here we would see only the leakages. As an approximation, the sequence impedances of synchronous machines are approximated as follows:

**Positive Sequence:** $X^1 = X_d^*$ or $X_d'$ or $X_d$, depending on analysis (sub-transient, transient or steady state.)

**Negative Sequence:** $X^2 = X_d^\prime\prime$, this is approximately correct.

**Zero Sequence:** $X^0 = X_f$, where $X_f$ is the leakage reactance.

### 10.3.4 Sequence Impedances of Transformers

We will study the per-phase parameters of the equivalent circuit of the transformer. The shunt (magnetization and core losses representation,) are usually neglected since they have a high impedance and they represent less than 1% of the rated transformer power. Also, since the transformer has no rotating fields or components, all the sequence impedances are the same (though sometimes the zero sequence current does not have a path to flow.) Hence we have: $Z^0 = Z^1 = Z^2 = Z_f$.

In a balanced delta connection and also in a 3-wire Y-connection, there will be no zero sequence currents outside the transformer; hence, these impedances will be represented with an open circuit (infinite zero-sequence impedance.) Recall that it is the convention to connect $Y - \Delta$ and $\Delta - Y$ transformers so that the high voltage side will lead the low voltage side by 30° (this is for the normal, or positive-sequence voltages.) For the negative-sequence voltages, this phase shift would be $-30°$. The zero-sequence impedance will depend on the connection of the two sides of the transformer. These are shown schematically below, for detailed diagrams, see page 413 of the book.

**Four-wire Ys:**

```
\[
\begin{align*}
\text{a} & \quad \text{g} \\
\text{a'} & \quad \\
\end{align*}
\]
```
Four-wire Y-Y:

N.B. The neutral (fourth wire) impedance to ground is very important since it appears with a factor of "3" times \( Z_n \). (This is due to the fact that if \( I_0 \) flows, then the neutral current will be \( I_n = 3I_0 \).)

**Example 10.3**

A balanced three phase voltage of 100-V line to neutral is applied to a balanced Y-connected load with ungrounded neutral. The three-phase load consists of three mutually coupled reactances. Each phase has a series reactance of \( Z_s = j12\Omega \), and the mutual coupling between phases is \( Z_m = j4\Omega \).

(a) Determine the line currents by mesh analysis without using symmetrical components.
(b) Determine the line currents using symmetrical components.
(a) Using KVL to the two meshes we have:

\[ Z_s I_a + Z_m I_b - Z_s I_b - Z_m I_a = V_a - V_b = |V_L| \angle \pi / 6 \]
\[ Z_s I_a + Z_m I_c - Z_s I_c - Z_m I_b = V_b - V_c = |V_L| \angle -\pi / 2 \]

And from KCL we have:

\[ I_a + I_b + I_c = 0 \]

Writing these three equations in matrix form:

\[
\begin{pmatrix}
(Z_s - Z_m) & -(Z_s - Z_m) & 0 \\
0 & (Z_s - Z_m) & -(Z_s - Z_m) \\
1 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
I_a \\
I_b \\
I_c
\end{pmatrix}
= 
\begin{pmatrix}
|V_L| \angle \pi / 6 \\
|V_L| \angle -\pi / 2 \\
0
\end{pmatrix}
\]

Or in compact form: \( Z_{\text{mesh}} I_{abc} = V_{\text{mesh}} \). Solving this equation using matrix inversion:

\[ I_{abc} = Z_{\text{mesh}}^{-1} V_{\text{mesh}} \]

The following Matlab program performs these operations:

```matlab
disp('(a) Solution by mesh analysis:');
Zs=j*12; Zm=j*4; Va=100; VL=sqrt(3)*Va;
Z=[(Zs-Zm)    -(Zs-Zm)       0 \\
    0        (Zs-Zm)     -(Zs-Zm) \\
    1           1          1]
V=[VL*cos(pi/6)+j*VL*sin(pi/6) \\
   VL*cos(-pi/2)+j*VL*sin(-pi/2) \\
   0]
Y=inv(Z);
Iabc=Y*V;
Iabcp=[abs(Iabc), angle(Iabc)*180/pi]
disp(Iabcp);
```

12.5000  (a) Solution by mesh analysis:  
12.5001 Iabcp =  
12.5002 12.5000 -90.0000  
12.5003 12.5000 150.0000  
12.5004 12.5000  30.0000

(b) Using symmetrical components we have:

\[ V_{012} = Z_{012} I_{012} \]

where:

\[ V_{012} = \begin{pmatrix} 0 \\ V_a \\ 0 \end{pmatrix} \]

and from past equations (page 7 with no \( Z_n \),) we have:

\[
Z_{012} = \begin{pmatrix}
Z_s + 2Z_m & 0 & 0 \\
0 & Z_s - Z_m & 0 \\
0 & 0 & Z_s - Z_m
\end{pmatrix}
\]

And now we have: \( I_{012} = [Z_{012}]^{-1} V_{012} \). This is done using the Matlab program below:
Example 10.4:

A three-phase unbalanced source with the following phase to neutral voltages

\[
V_{abc} = \begin{bmatrix}
200 \angle 25^\circ \\
100 \angle -155^\circ \\
80 \angle 100^\circ 
\end{bmatrix}
\]

is applied to the circuit shown. The load series impedance per phase is \( Z_s = 8 + j24 \) and the mutual impedance between phases is \( Z_m = j4 \). The load and source neutrals are solidly grounded. Determine:

(a) The load sequence impedance matrix \( Z_{012}^{\text{abc}} = A^{-1}Z_{abc}A \).
(b) The symmetrical components of voltage.
(c) The symmetrical components of current.
(d) The load phase currents.
(e) The complex power delivered to the load in terms of symmetrical components.
(f) The complex power delivered to the load in terms of the power per phase.

The following Matlab program is the solution:

```matlab
clear
Vabc = [200 25 100 -155 80 100];
Zabc = [8+j*24 j*4 j*4 j*4 8+j*24 j*4 j*4 j*4 8+j*24];
Z012 = zabc2sc(Zabc);
V012 = abc2sc(Vabc);
V012p = rec2pol(V012);
I012 = inv(Z012)*V012p;
I012p = rec2pol(I012);
Iabc = sc2abc(I012);
```

Which is identical with the answer obtained in part (a).
\[
\begin{align*}
I_{abcp} &= \text{rec2pol}(I_{abc}) \\
S_{3\phi_{sc}} &= 3 \cdot (V_{012}.) \cdot \text{conj}(I_{012}) \\
V_{abc} &= V_{abc}(\cdot, 1) \cdot \text{cos}(\pi/180 \cdot V_{abc}(\cdot, 2)) + j \cdot \text{sin}(\pi/180 \cdot V_{abc}(\cdot, 2)) \\
S_{3\phi} &= (V_{abc}.) \cdot \text{conj}(I_{abc}) \\
\end{align*}
\]

\[
Z_{012} =
\begin{bmatrix}
8.0000 + 32.0000i & 0 + 0.0000i & 0 + 0.0000i \\
0.0000 + 0.0000i & 8.0000 + 20.0000i & -0.0000 + 0.0000i \\
0 + 0.0000i & 0.0000 - 0.0000i & 8.0000 + 20.0000i
\end{bmatrix}
\]

\[
V_{012p} =
\begin{bmatrix}
47.7739 & 57.6268 \\
112.7841 & -0.0331 \\
61.6231 & 45.8825
\end{bmatrix}
\]

\[
I_{012p} =
\begin{bmatrix}
1.4484 & -18.3369 \\
5.2359 & -68.2317 \\
2.8608 & -22.3161
\end{bmatrix}
\]

\[
I_{abcp} =
\begin{bmatrix}
8.7507 & -47.0439 \\
5.2292 & 143.2451 \\
3.0280 & 39.0675
\end{bmatrix}
\]

\[
S_{3\phi_{sc}} =
\begin{bmatrix}
9.0471 \times 10^{02} + 2.3373 \times 10^{03}i
\end{bmatrix}
\]

\[
S_{3\phi} =
\begin{bmatrix}
9.0471 \times 10^{02} + 2.3373 \times 10^{03}i
\end{bmatrix}
\]